Some insights on the study of Dynamical Systems

Arturo Pérez Roncero

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1 Intruduction

Chaos theory is a branch of science that studies the nature of a specific kind of dynamic systems (functions that represent systems that change over time) that are especially susceptible to slight variations in their original state. Because of my interest in this field, I happened to encounter a peculiar type of dynamic systems that I had not seen before while surfing the Internet. The finding was made by the YouTube channel, CodeParade[1], which posted a video on YouTube on the subject. These dynamic systems consisted of apparently unpredictable and chaotic patterns, which were constantly evolving, and from now on, to make my work more understandable, I will call them "undefined dynamic systems".

On my first contact with them, I was unable to determine their exact nature, as they were nothing like any of the dynamic systems I had seen before, and there was no previously published study on them other than the above-mentioned video. This, however, includes enough information to start an investigation about their real nature. This paper will do just that: a thorough search for the hidden truth of these dynamic systems. For this purpose, these systems will be compared with other better known and studied mathematical phenomena, and then a simulator will be developed from scratch, in order to understand their behavior in greater depth.

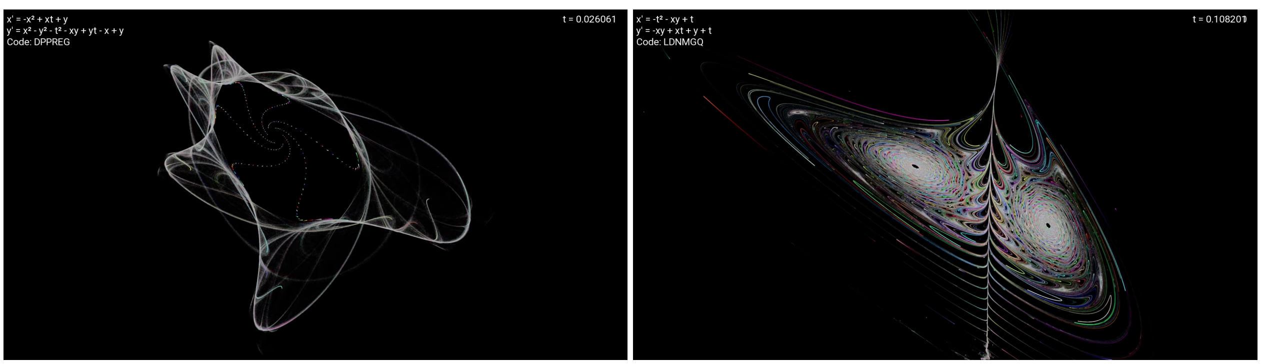


Figure 1: Snapshots from the video.

[1]https://www.youtube.com/watch?v=fDSIRXmnVvk&t=

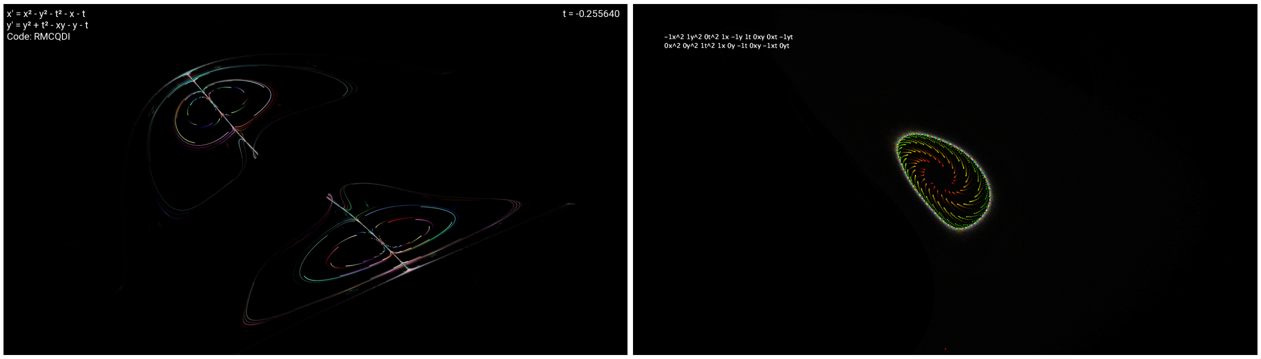


Figure 2: Snapshots from the video.

2 Differential equations and attractors

When I asked myself what these dynamic systems were all about, the first thing that came to mind was the Lorenz Attractor. The Lorenz Attractor is a system of 3 partial differential equations, developed by Edward Lorenz in the 20th century. It is an attempt to derive a purely physical concept using the language of mathematics. More specifically, the representation of the behavior of the air according to a certain atmospheric convection model. Perhaps the most famous dynamic system in Chaos Theory, and the emblem of deterministic chaos, is understandable why it was the first solution that occurred to me. However, to find out if the undefined dynamic systems and the Lorenz Attractor really work under the same concepts, and do not just have a simple visual similarity, we will have to investigate it further, starting with differential equations.

A differential equation is any equality that is somehow able to relate a given function to any of its derivatives. By having more than one variable per equation, we know that they are a system of partial differential equations.

What a differential equation shows us can be, for example, the change of position of a function with respect to its original position on the corresponding axle. For example, $\frac{dx}{dt} = 2$ would translate as a one-dimensional function changing position on that axis at a rate of two units of space for each unit of time. This makes representing them with a program extremely simple, as a computer can easily repeat a very simple mathematical task thousands of times per second, even though, as chaos theory tells us, its behaviour may in some cases be stochastic or unpredictable.

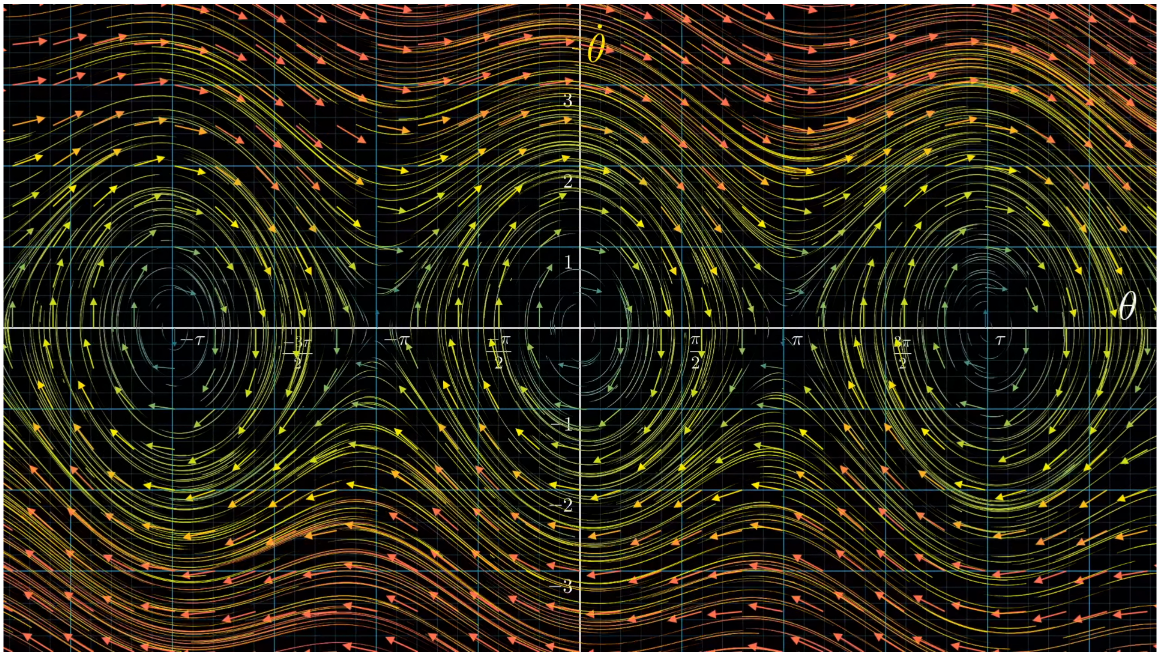


Figure 3: Vector field for the movement of a pendulum. The represented vectors show the direction and magnitude with which a pendulum would move in that state (represented in the plane as a point.[2]

Differential equations are capable of describing an infinite number of real physical phenomena, if not all of them. From the movement of a pendulum (figure above), to changes in temperature in a three-dimensional object, differential equations have allowed us to represent nature mathematically in a simple and easily studied way. However, the study of differential equations is very complex, and the simple fact of knowing whether or not they have a solution is a major challenge. A clear example is the reduced problem of the three bodies, a model that represents the behavior of three bodies that interact gravitationally by means of a system of differential equations. With this model the possible orbits of a very low mass object can be plotted around two other bodies. Most of these orbits are complex and difficult to predict, but there are a number of solutions to the equations which result in stable and periodic orbits, or halo orbits. It is practically impossible for current mathematics to find these solutions analytically, given their complexity (see equations below), and it is necessary to resort to computer simulations in order to find them approximately.

$$\begin{cases} \ddot{x} = 2\dot{y} + x - \frac{1-\mu}{r_{13}^3}(x+\mu) - \frac{\mu}{r_{23}^3}(x-1+\mu) \\ \ddot{y} = -2\dot{x} + y - \frac{1-\mu}{r_{13}^3}y - \frac{\mu}{r_{23}^3}y \\ \ddot{z} = -\frac{1-\mu}{r_{13}^3}z - \frac{\mu}{r_{23}^3}z \end{cases} \tag{1}$$

Before studying the solution of a differential equation, several subtypes within them should be highlighted. First there are the linear differential equations, which get their name from the fact that their unknown x

[2] https://www.youtube.com/watch?v=p_di4Zn4wz4

is always raised to the unit. All their solutions are linear combinations of the others, and can be found by using formulas, given their simplicity, or even by integrating both parts of the equation. Their general form is as follows:

$$\frac{dx}{dt} = f(t) * x + g(t)$$

The general solution to any differential equation is that, or those functions which by substituting them into that equation make it an identity. To solve this linear differential equation the following will be done:

$$\frac{dx}{dt} = 2t \tag{2}$$

$$\int dx = \int 2t dt \tag{3}$$

$$x = 2 * \frac{t^2}{2} + c \tag{4}$$

$$x = t^2 + c \tag{5}$$

Where c is the necessary arbitrary constant, as it is a general solution.

Apart from linear equations, there are non-linear ones, of much greater complexity, whose solutions cannot usually be found by analytical methods, as explained above. Like the problem of the three bodies, there is the possibility of studying very specific cases, where there are symmetries that allow simplification, but these cases are rare.

Although there are methods for finding these solutions in specific types of differential equations, they are not useful for the Lorentz attractor, since it is a system of three equations with more than one variable each (partial), increasing the complexity considerably, making it an example of a non-linear differential equation.

A basic quality of the Lorentz Attractor is that of changing its appearance depending on the value of the constants σ, β and ρ. A slight variation in these may cause the system to change from convergent to divergent. With a simple computer program we can simulate the result of the attractor with the values: $\sigma = 10$, $\beta = \frac{8}{3}$ and $\rho = 28$, and the equations:

$$\frac{dx}{dt} = \sigma(y - x) \tag{6}$$

$$\frac{dy}{dt} = x(\rho - z) - y \tag{7}$$

$$\frac{dz}{dt} = xy - \beta z \tag{8}$$

Where x represents the convection rate, y the horizontal temperature variation, and z the vertical temperature variation.

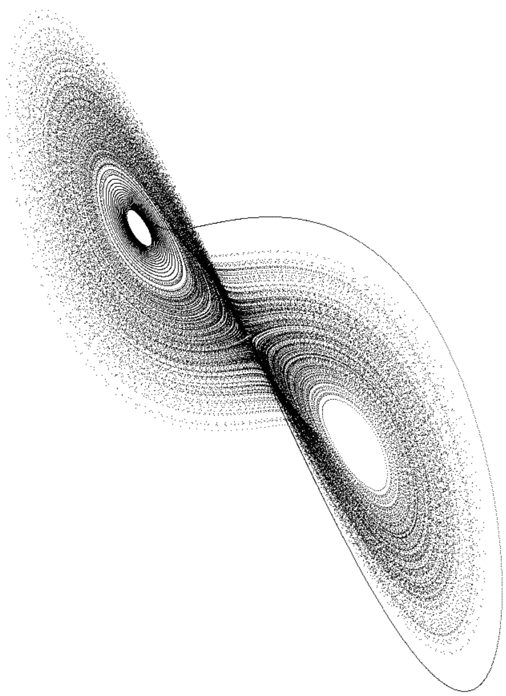
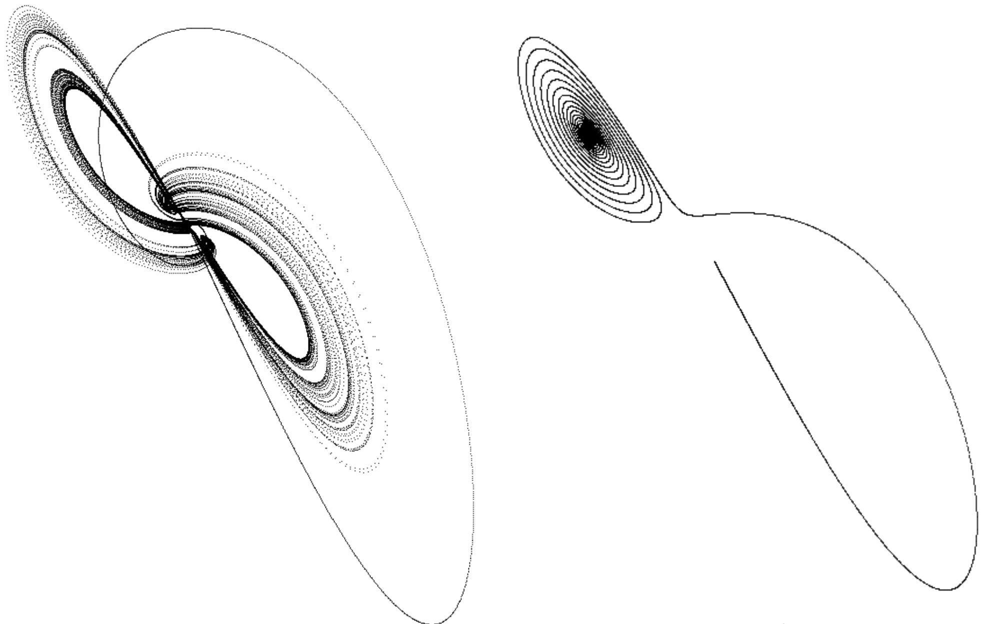


Figure 4: Lorenz Attractor. $\sigma = 10$, $\beta = \frac{8}{3}$ y $\rho = 28$. Simulated with self-developed program.

```
float a = 10;
float b = 28;
float c = 8.0/3.0;

float x,y,z,dt,dif,col;

void setup(){
  background(0);
  fullScreen();
  x=.01;
  dif=0.0000001;
}

void draw(){
  translate(width/2,height/2);

  x+=a*(y-x)*dt;
  y+=(x*(b-z)-y)*dt;
  z+=(x*y-c*z)*dt;

  dt+=dif;

  stroke(255);
  point(16*x,16*y);
}
```

Listing 1: Lorenz attractor simulator.

Sin embargo, en cuanto cambiamos σ, β y ρ, los resultados empiezan a variar:



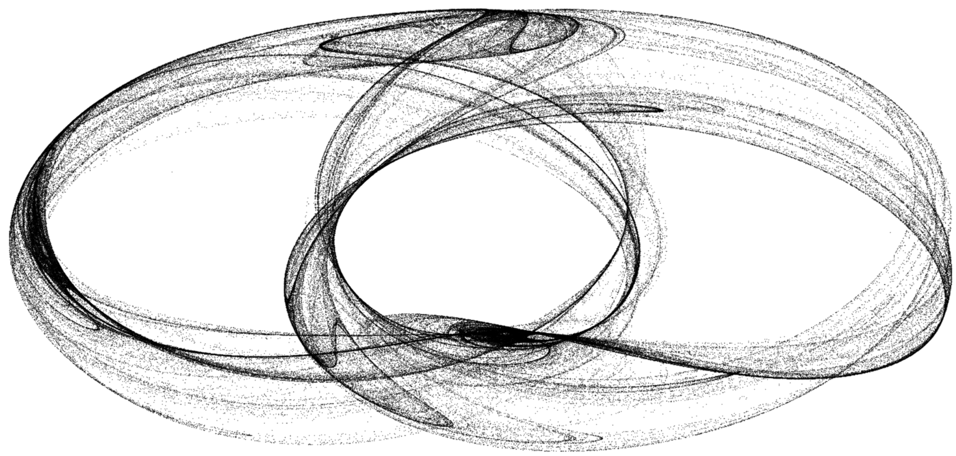
(b) $\sigma = 28$, $\beta = \frac{2}{3}$ y $\rho = 10$.

(a) $\sigma = 10$, $\beta = \frac{2}{3}$ y $\rho = 28$.

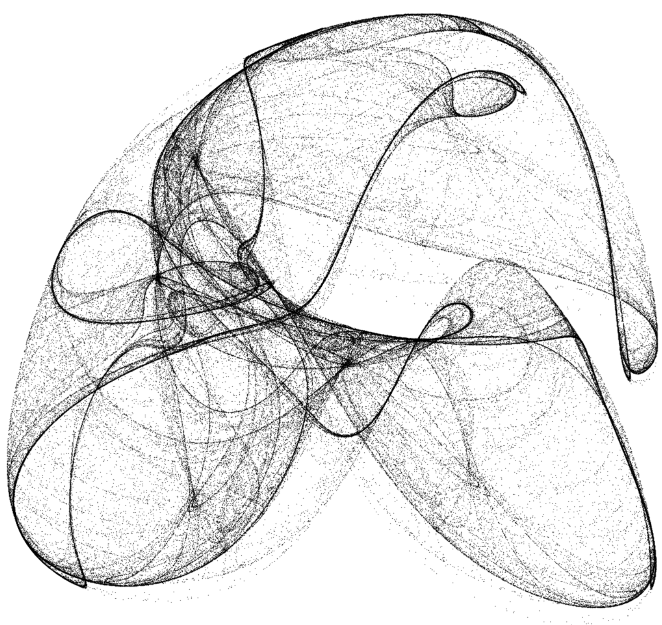
Figure 5: Simulated attractors with self-developed program.

Other attractors, such as Clifford's or Peter de Jong's, share many similarities with Lorenz's, as do some of our indeterminate dynamic systems. As we have seen, they are capable of radically changing their appearance by varying very slightly some of their parameters, behavior that we have observed in indeterminate systems as well. However, there are some obvious differences that we have overlooked:

- Indeterminate dynamic systems show a different type of activity. In the Lorenz system we have a single point that is updated with the new position dictated by the differential equations. However, in our dynamic systems we have a much more complex figure than a point, being normally a convergent, or divergent, curve that creates convoluted patterns, which is updated in its entirety by each unit of time elapsed.



(a) Clifford Attractor.



(b) Peter de Jong Attractor.

Figure 6: Simulated attractors with self-developed program.

- The video (Figure 6) shows the system of equations that is being represented at that time. Here we see functions like $x' = -x^2 + xt + y$; $y' = x^2 - y^2 - t^2 - xy + yt - x + y$ (red square). This would be the proper notation to represent a system of differential equations, in which the derivative of a function is related to the function in question. That is, it shows us the rate at which each of the coordinates changes for each unit of time. However, when explaining the procedure used to manage to represent the indeterminate dynamic system, it does not use differential equations, it uses an iteration process.

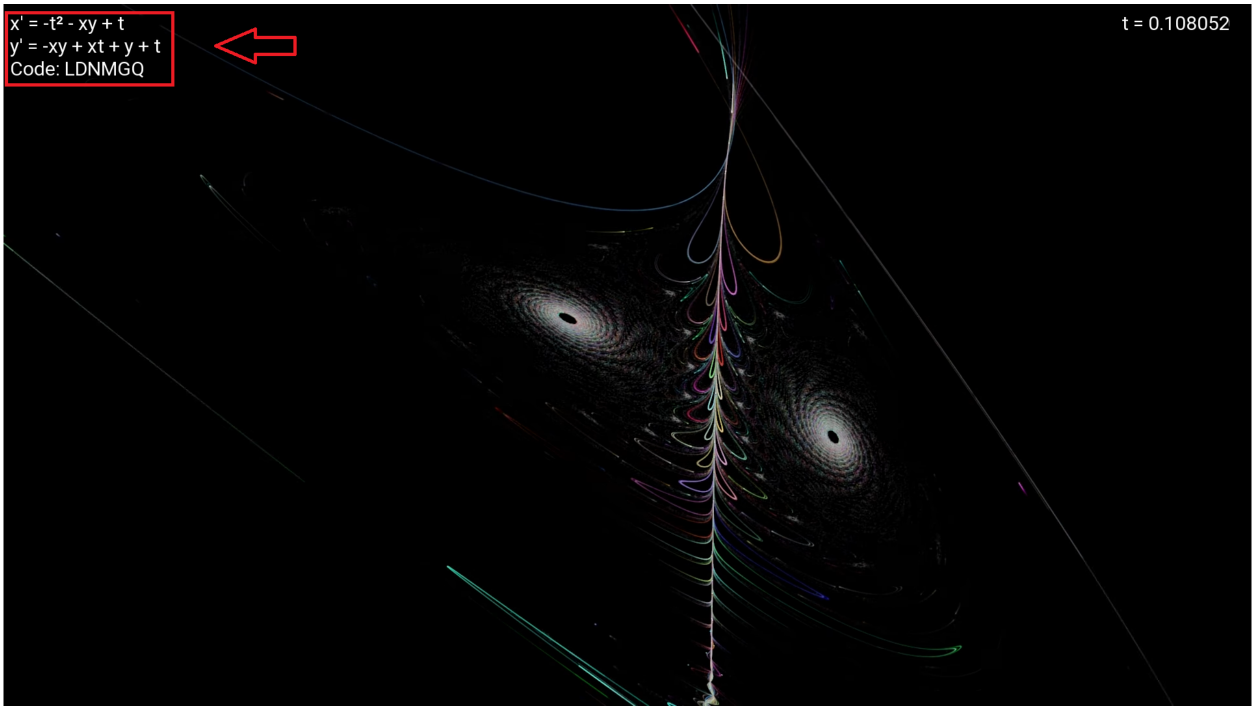


Figure 7: Snapshots from the video.

3 Iterative sets

Realizing that it was impossible for these indeterminate dynamic systems to be the product of differential equations, I began to look for other solutions. Immediately, the Mandelbrot set came to mind.

Iteration is a mathematical process that produces iterative sequences from one or several functions. It consists of introducing the solution to the function at a certain point as an input for the function itself, which must be solved again, to give in turn a new solution. By repeating this process, a sequence of solutions is obtained that can create very complex patterns from apparently simple functions, as is the case with the Mandelbrot set.

In the Mandelbrot set the following function is taken: $f_c(z) = z^2 + c$. Where z is a complex number with the form $a + bi$, and c is the previous term in the progression. This will give us back a succession of complex numbers, which we can represent in a two-dimensional plane, taking the real part as the coordinates on the x-axis and the imaginary part as those on the y-axis. This will give us back a pattern that will depend on the starting point in which we start to iterate.

In some cases it is divergent, and the solutions tend to infinity; and in others, it converges, and the solutions tend towards the origin of ordinates.

If we calculate the sequences generated by this method from the position of all the points on a two-dimensional plane, and we colour in white the points that generate convergent sequences, and in black those that generate divergent sequences, we will obtain the Mandelbrot set as we know it:

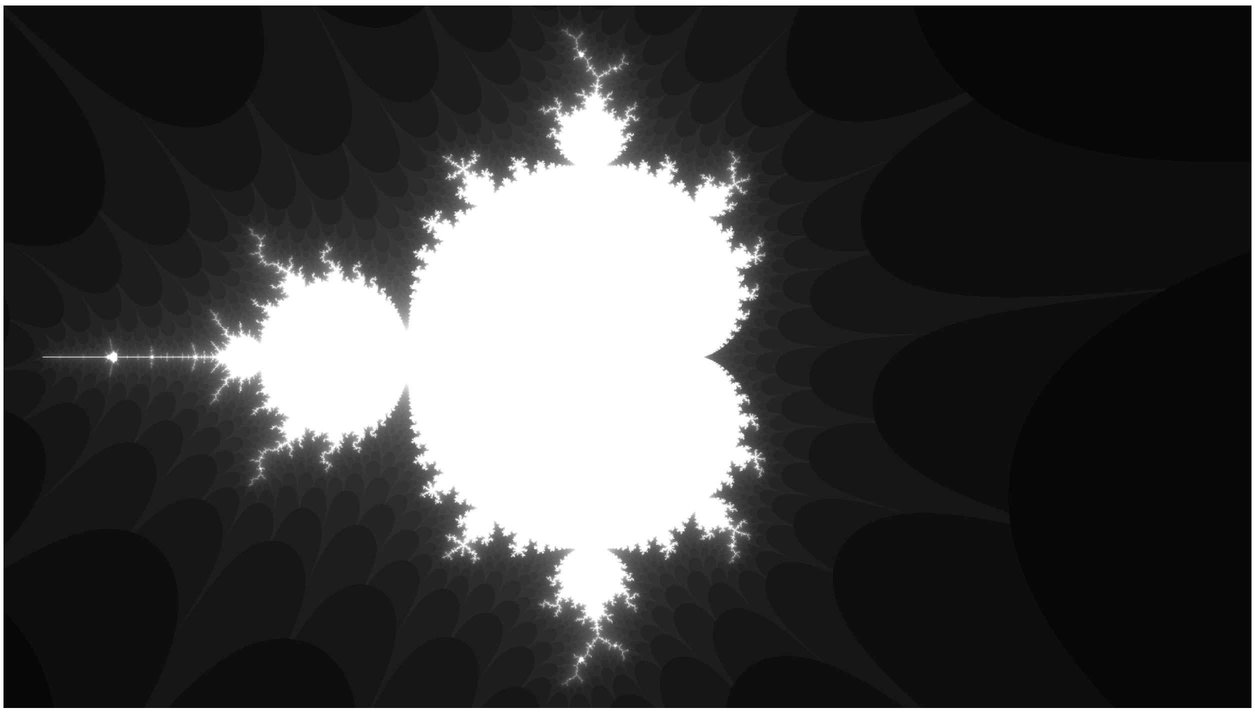


Figure 8: Mandelbrot Set simulated my myself.

The concept used to obtain the Mandelbrot set is very similar to that used in undetermined dynamic systems. However, it is clear that none of the results have any similarity with what we are looking for. There are several reasons for this:

- First of all, the iterated function to find the Mandelbrot set is much simpler than those of the indeterminate dynamic systems, being this a possible cause of the difference between both sequences.
- Secondly, the Mandelbrot set is not a dynamic system, as it does not change as a function of time, but should still resemble the snapshot of an indeterminate system in that case.
- Finally, and perhaps most importantly, the system of representation of the two iterations is very

different. The criterion for representing the Mandelbrot set is the tendency of the iterated sequence at each point, coloring that point in one shade or another. In indeterminate dynamic systems we have a system of parametric equations that represent a succession in the plane. Both are iterations, since in the indeterminate system the value calculated in the previous result is also introduced as the value of the variables, and most differential equations are still iterations of the same mathematical operation; however, while in the Mandelbrot set it is represented whether the incisive value put into iteration results in convergence or divergence, in the indeterminate system what is represented is the iteration as such. Therefore, it is reasonable that both figures do not resemble each other at all.

4 Parametric equations

Parametric equations are a type of equation in which the value of the positions in each of the dimensions used is expressed separately. These expressions define the position of the resulting point in the totality of time, meaning that it is not necessary to iterate, as in differential equations, and simply calculate the result for all desired values. Let us take the following system of parametric equations:

$$x = \cos t \tag{9}$$

$$y = \sin t \tag{10}$$

This will give us back a function in the form of the unitary circle. The position on the X axis will be given by the first equation, and the position on the Y, by the second.

Parametric equations are widely used to describe physical behaviour, such as uniformly accelerated motion or any of the orbits that one body may take around another (conic curves). It is easier to study their results than in the differential equations.

Thanks to the information given in the video about how indeterminate dynamic systems are generated, we can conclude that these are formed by parametric equations in which a t variable is included to represent time. This means that to find the evolution of the system along time (t), it is not necessary to iterate, and we only have to calculate the result for the different values of t desired. Iteration is only carried out to calculate the system itself, and not to find its evolution.

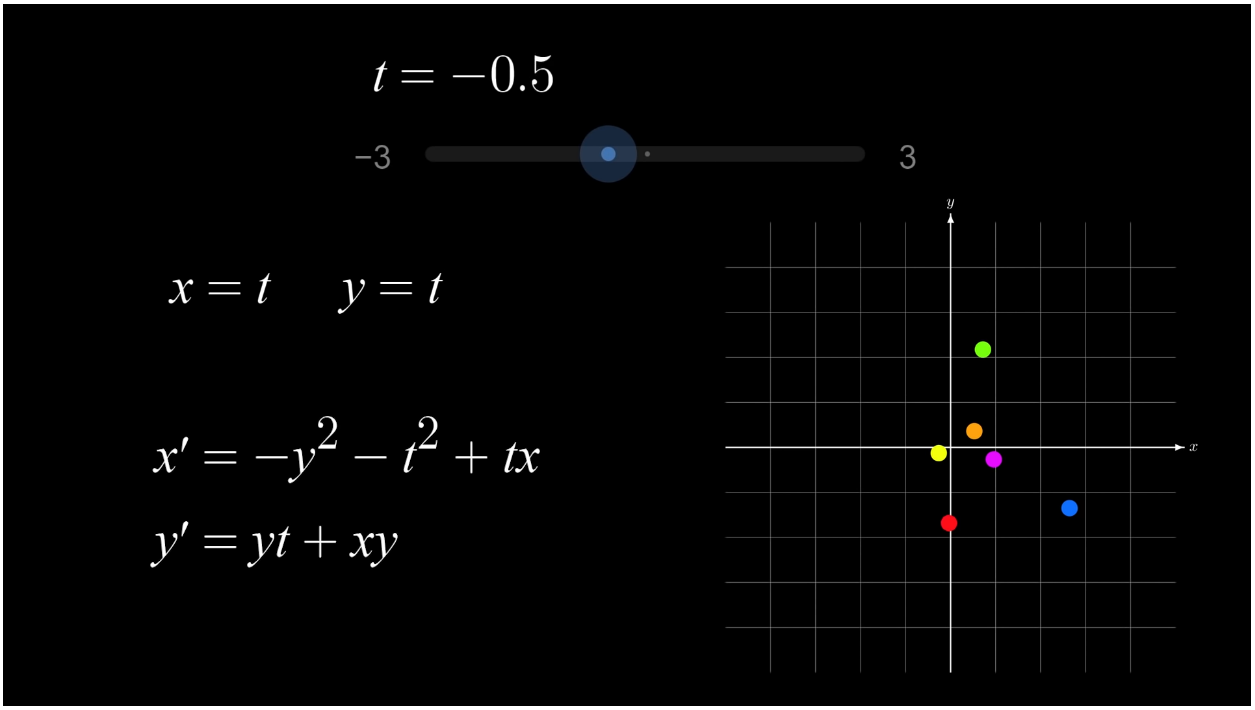


Figure 9: Snapshots from the video.

5 Study and Conclusion

Finally, we can conclude that indeterminate dynamic systems are, in fact, dynamic systems formed from sequences represented in the plane and generated by the iteration of systems of random two-dimensional parametric equations.

In other words, from a random system of two parametric equations (where the unknown can be found in monomials of all types, roots, exponents, trigonometric functions, etc...) we introduce a variable t that represents time. We give any value to all the unknowns we have (in this case x and y). The result that we obtain in each one of the equations we introduce it again as values of the unknowns in those equations, starting an iteration process that we repeat until what we consider sufficient. This will generate a succession that we will represent (in our case). Varying slightly the t that we have mentioned before, the sequence will start to vary in form, giving rise to our dynamic system.

Immediately we start to see dynamic systems like the one in the video:

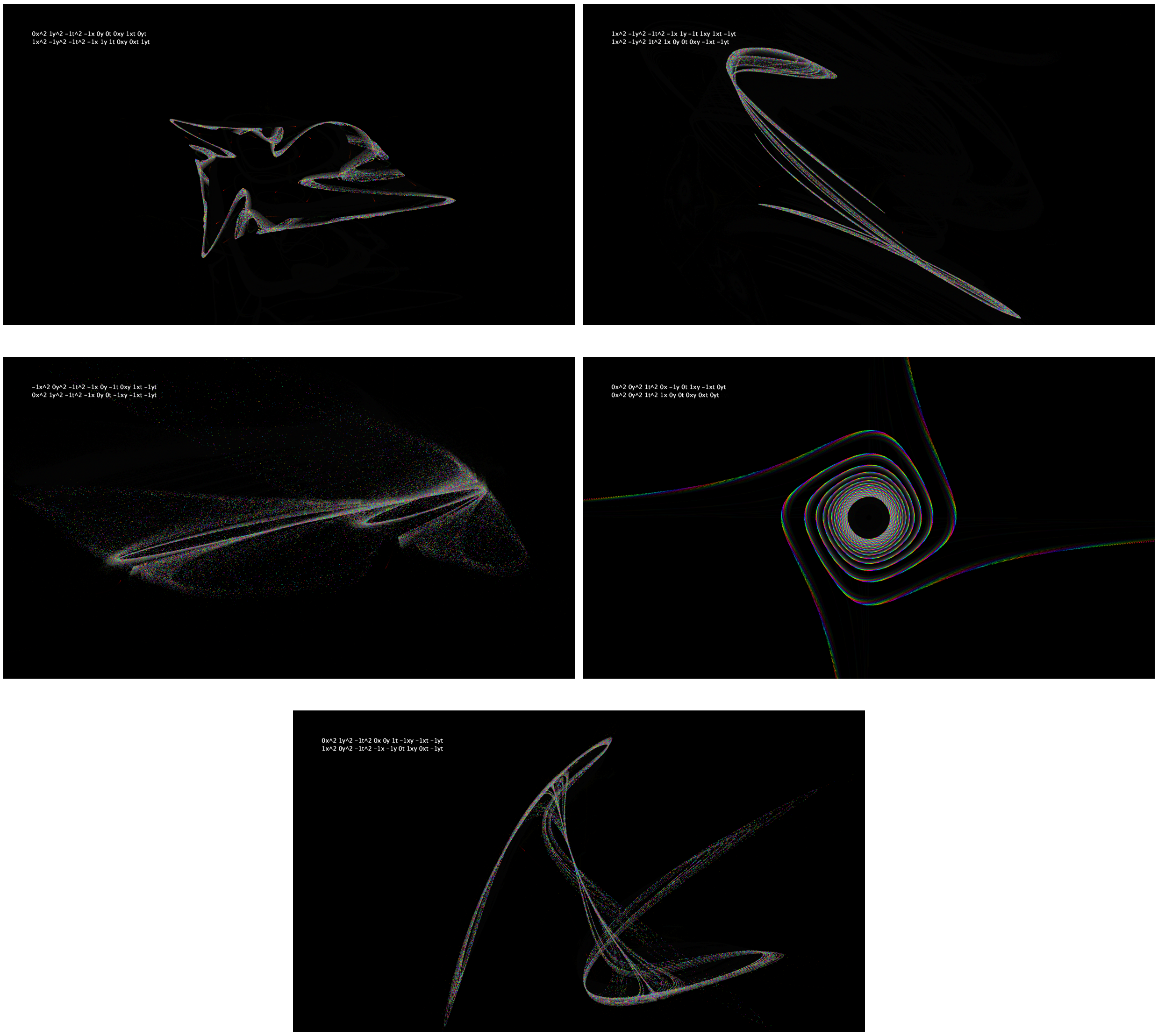


Figure 10: Simulator's results.

These dynamic systems are extremely complex when trying to predict their behavior, given their chaotic and variable nature. Changing the value of a single constant in either equation results in a radical change in the development of the system. For all these reasons, it is very complicated to classify them, only one of their qualities being evident at first sight: convergence. If the system converges within a reasonably small domain of t to be able to study it, the program previously designed to simulate it would be very useful when trying to find the moment in which the system converges.

5.1 Possible behaviours

5.1.1 Black holes

Many of these dynamic systems create very characteristic patterns when converging, a very common example being image number 4 of the latter system. They begin to close (or open, depending on the sign of the elements) over specific areas of the plane ("black holes") and then concentrate on a single point. After that moment, the behaviours vary a lot, and it has been impossible for me to find a recognizable pattern, something that reaffirms that they are really chaotic-deterministic dynamic systems.

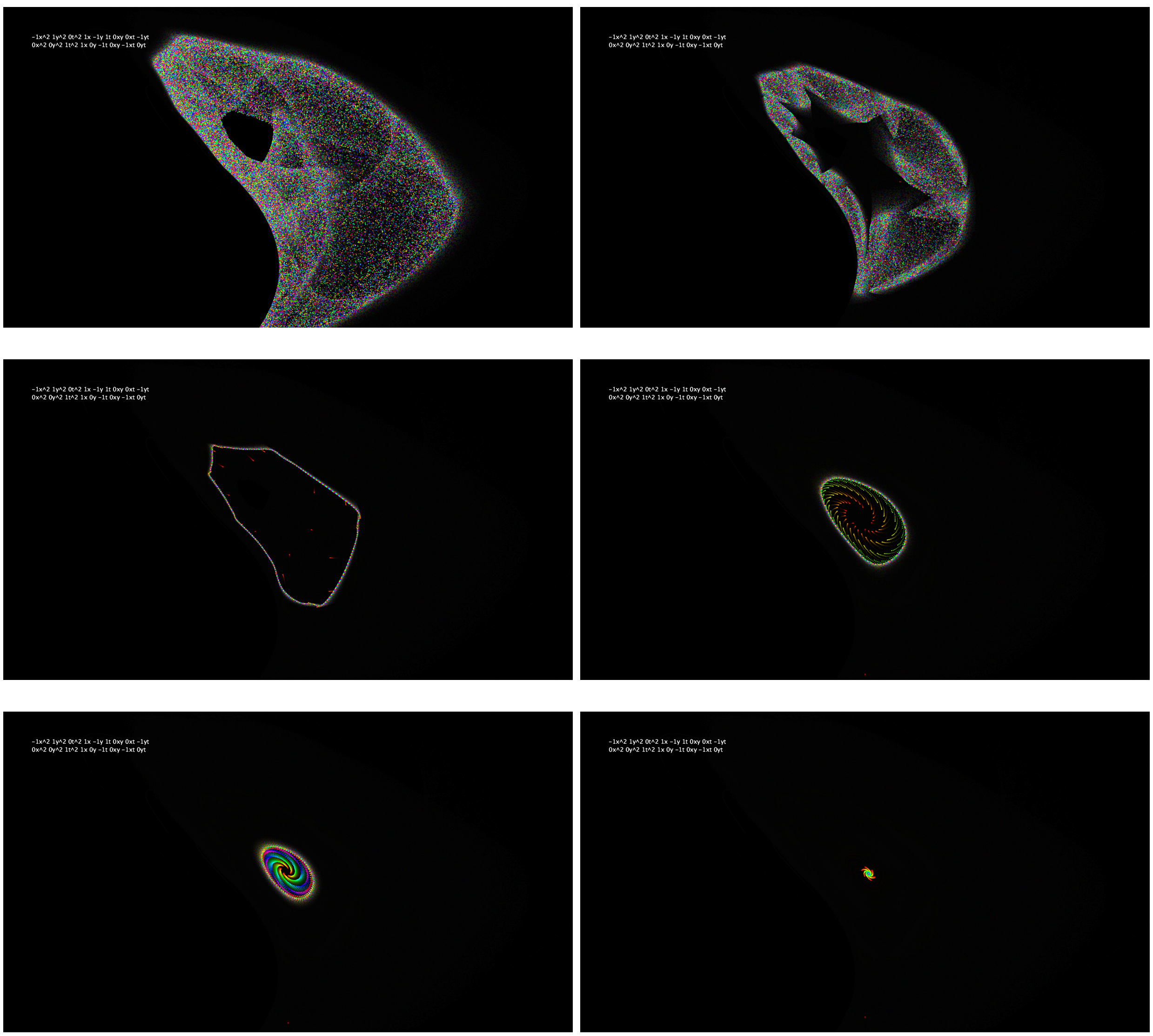


Figure 11: Simulator's results.

5.1.2 Single and multiple spirals

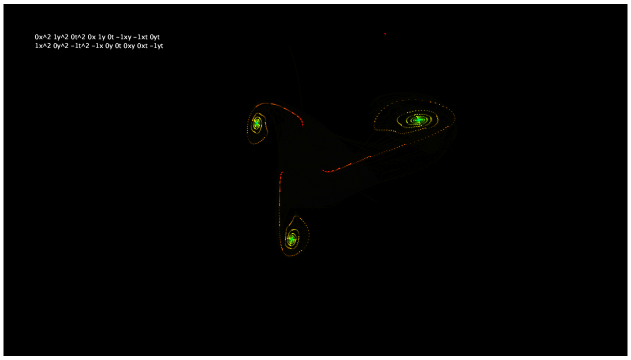


Figure 12: Rectangle

In most cases, after a convergence is observed, a spiral is generated that flows from or to that point of convergence. Sometimes several points of convergence seem to coexist at different points in the plane, giving rise to "multiple spirals". The existence of these suggests the possibility of more than one point of convergence, which is possible because the system is in a two-dimensional plane.

5.1.3 Linear results

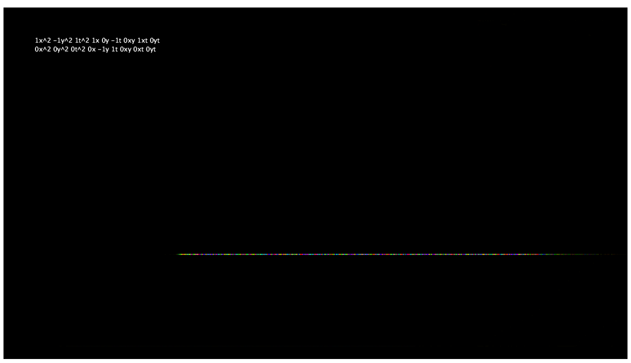


Figure 13: Rectangle

As in one of the previous examples, the area of convergence to focus sometimes on a line instead of a point. This may be because convergence exists in only one of the two equations, and while on one axis it converges, on the other it does not.

5.1.4 Trees

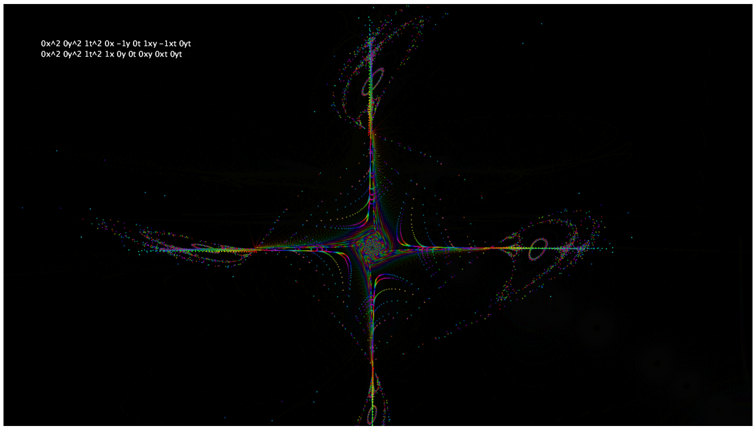


Figure 14: Rectangle

"Trees" are often irregular and volatile areas where many points come together to form circular patterns and/or branches, reminiscent of the structure of a tree. They are most likely the result of black holes, spirals, and linear results, resulting in a much more complex pattern.

5.1.5 Intermittences

Finally, sometimes systems acquire an intermittent behavior in which they vary extremely quickly between two or more of the above-mentioned behaviors.

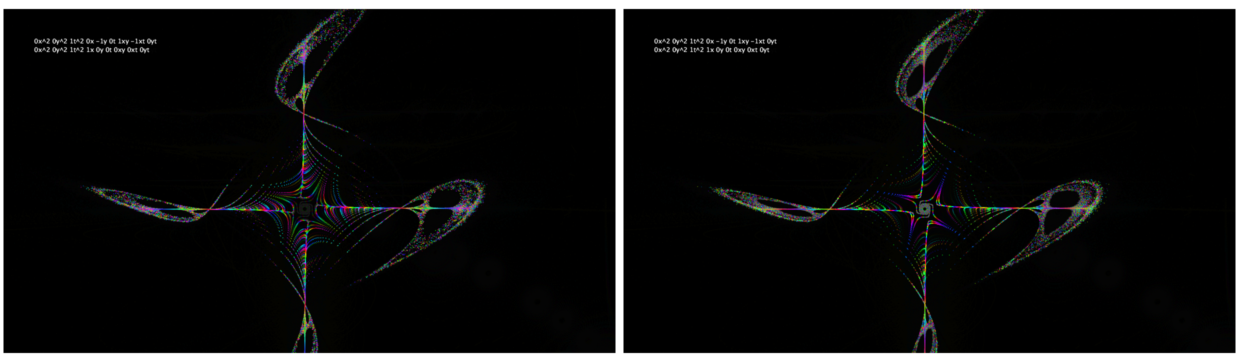


Figure 15: Simulator's results.

In this example, a "black hole" in the middle of the system appears and disappears intermittently from the function. This behavior may be due to convergence processes (like the black hole in the figure above) that develop in such a short time that the jumps made by the simulator in the value of t are too big to appreciate the growth or decrease of the resulting shapes, creating the illusion of an intermittence, when in fact they are extremely fast evolutions.

6 Bibliography

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